Composites for geotechnical engineers – an introduction

Author: Prof. Dr. habil. Heinz Konietzky
(TU Bergakademie Freiberg, Geotechnical Institute)

1 Preface ... 2
2 Basic mathematical mixture rules ... 4
3 Basic mixture rules for composites ... 5
4 Mixture rules under consideration of Possion’s ratio ... 9
5 Halpin-Tsai equation ... 10
6 Average density ... 10
7 Fibre reinforced materials ... 10
8 Simple examples ... 14
8.1 Characteristics of a layered roof ... 14
8.2 Characterstics of a composite support element ... 14
8.3 Characteristics of fibre reinforced concrete ... 16
9 References ... 18

1 Preface

A composite is a material which is produced by two or more different constituent materials. Composites play a vital role in material sciences, product development and any kind of engineering, but also in daily life. Fig. 1.1 provides an overview about composites from the point of view of material sciences. Fig. 1.2 illustrates some significant geometric and spatial characterzistics of composites influencing the material behaviour on the basis of fibres embedded in a matrix.

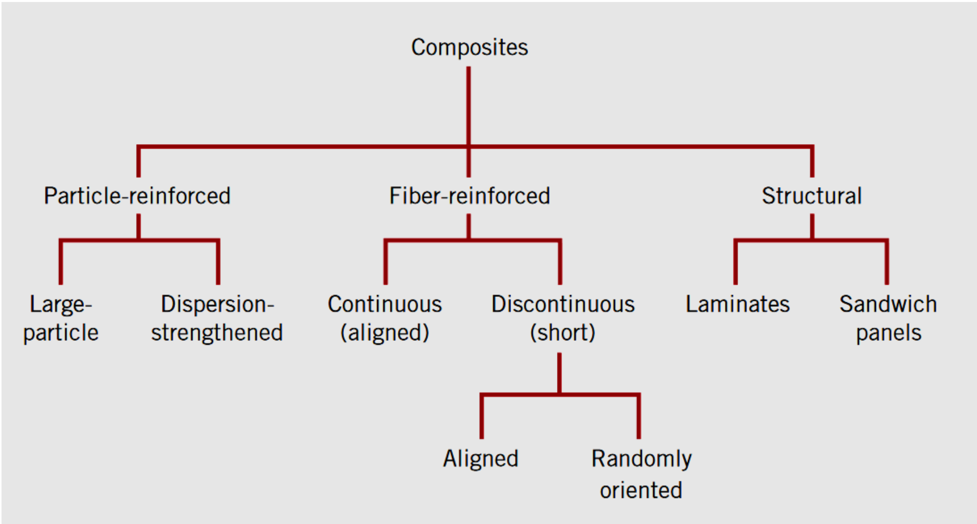

Fig. 1.1 Classification of composites (Callister & Rethwisch, 2009)

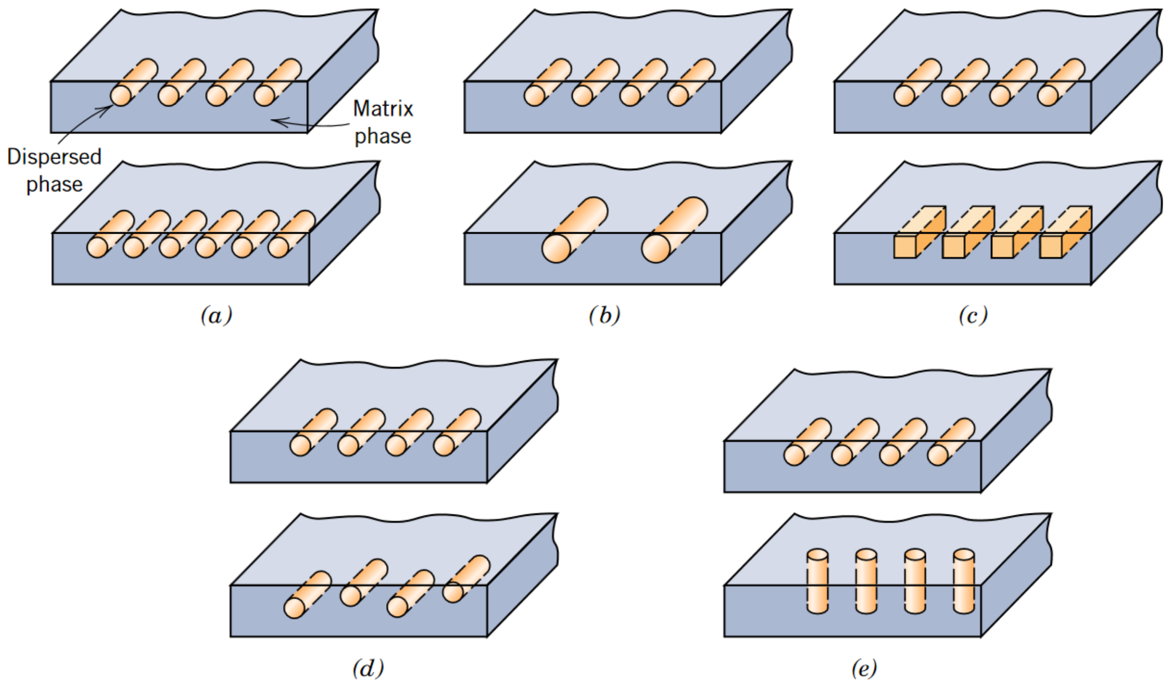

Fig. 1.2 Geometrical and spatial characteristics for particle and fibre reinforced composites: (a) concentration, (b) size, (c) shape, (d) distribution, (e) orientation (Callister & Rethwisch, 2009)

Typical applications of composites in geo-engineering are:

- Steel- or fibre-reinforced concrete or shotcrete
- Several layers of lining or support
- Masonry constructions
- Geotextile reinforced earth
- Bedding or schistosity

In addition, also the geological layering can be consideres as a large-scale composite; and at the micro-scale any type of rock is a composite in terms of pores and matrix as well as several minerals.

The behaviour of composites in general is quite complex and therefore nowadays numerical simulation techniques are used for predict or analyse the behaviour of such materials.

There are a few key questions in respect to the behaviour of composites in geo-engineering, for instance:

- How is the distribution of stress and strain in the diferent constituent materials under loading ?

- How is the overall stiffness and deformability of a composite as a whole ?

- How is the behaviour (incl. strength, failure etc.) at the interface between several constituent materials ?

This e-book chapter describes only the basic mixture rules for composites considering pure elastic behaviour and some practical applications.

2 Basic mathematical mixture rules

There are 6 different basic mixture rules in mathematics important for composites:

- Arithmetic mean

$$x_A = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{2.1}$$

- Weighted arithmetic mean

$$x_{WA} = \sum_{i=1}^{n} \varphi_i x_i \quad with \quad \sum_{i=1}^{n} \varphi_i = 1 \tag{2.2}$$

Where φ is the weighting factor

- Harmonic mean

$$x_H = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} \tag{2.3}$$

- Weigthed harmonic mean

$$x_{WH} = \frac{\sum_{i=1}^{n} \varphi_i}{\sum_{i=1}^{n} \frac{\varphi_i}{x_i}} \tag{2.4}$$

- Geometrical mean

$$x_G = \sqrt[n]{\prod_{i=1}^{n} x_i} \tag{2.5}$$

- Weighted geometrical mean

$$x_{WG} = \sqrt[\varpi]{\prod_{i=1}^{n} x_i^{\varpi_i}} \quad with \quad \varpi = \sum_{i=1}^{n} \varpi_i \tag{2.6}$$

It holds in general: $x_A \geq x_G \geq x_H$.

3 Basic mixture rules for composites

There are tow basic mixture rules for composites:

- Voigt mixture rule
- Reuss mixture rule

They consider only two elastic components represented by two springs (see Fig. 3.1 and 3.2). The Reuss model considers two springs A and B in series which experience the same stress (isostress mixing rule). The Voigt model considers two springs A and B in parallel which experience the same strain (isostrain mixing rule).

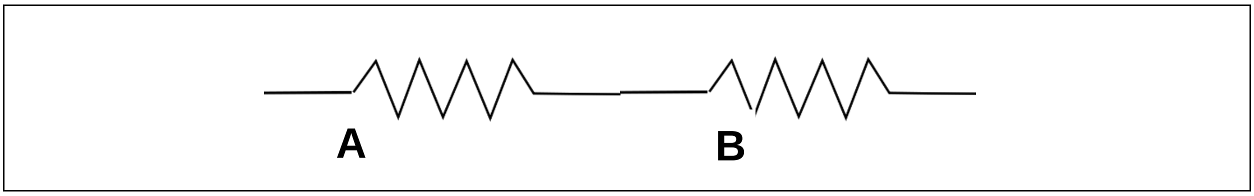

Fig. 3.1 Reuss mixture model

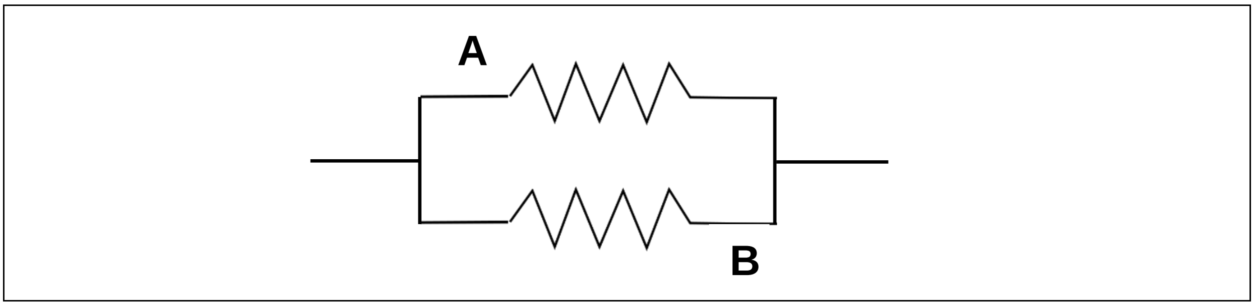

Fig. 3.2: Voigt mixture model

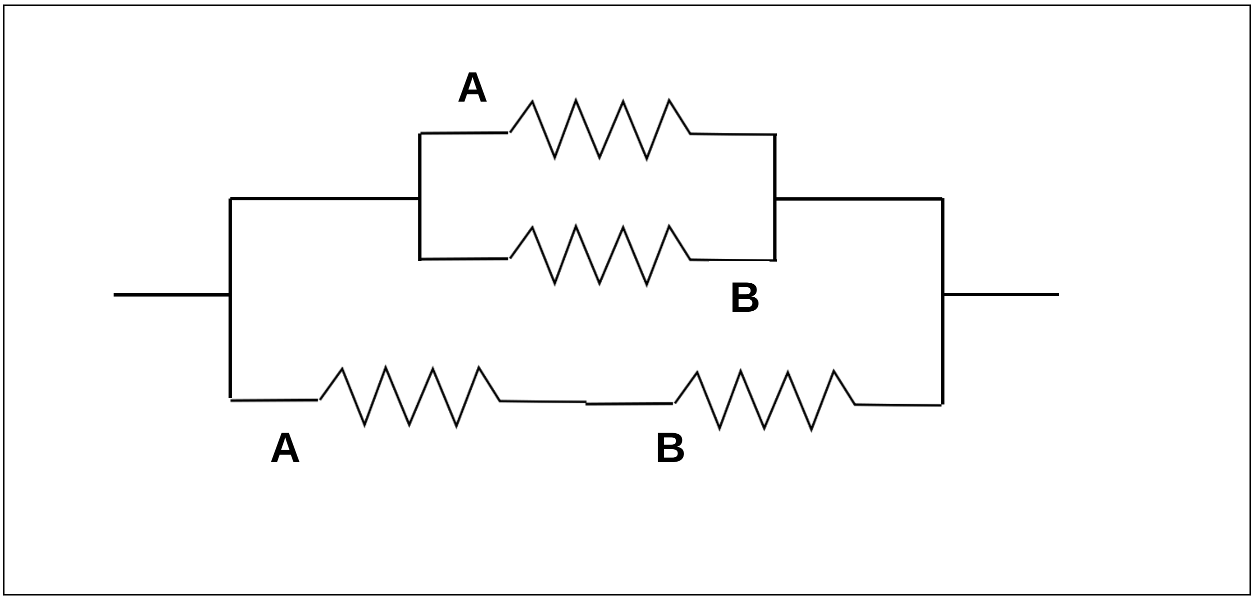

Fig. 3.3: Hybridized Voigt-Reuss mixture model

For the 1-dimensional Reuss model the following yields (always: $\varphi_A + \varphi_B = 1$):

$$\varepsilon_{total} = \varphi_A \varepsilon_A + \varphi_B \varepsilon_B \quad \text{and} \quad \sigma_{total} = \sigma_A = \sigma_B \tag{3.1}$$

$$E_{Reuss} = \frac{E_A E_B}{\varphi_A E_B + \varphi_B E_A} \tag{3.2}$$

For the 1-dimensional Voigt model the following yields:

$$\varepsilon_{total} = \varepsilon_A = \varepsilon_B \quad \text{and} \quad \sigma_{total} = \varphi_A \sigma_A + \varphi_B \sigma_B \tag{3.3}$$

$$E_{Voigt} = \varphi_A E_A + \varphi_B E_B \tag{3.4}$$

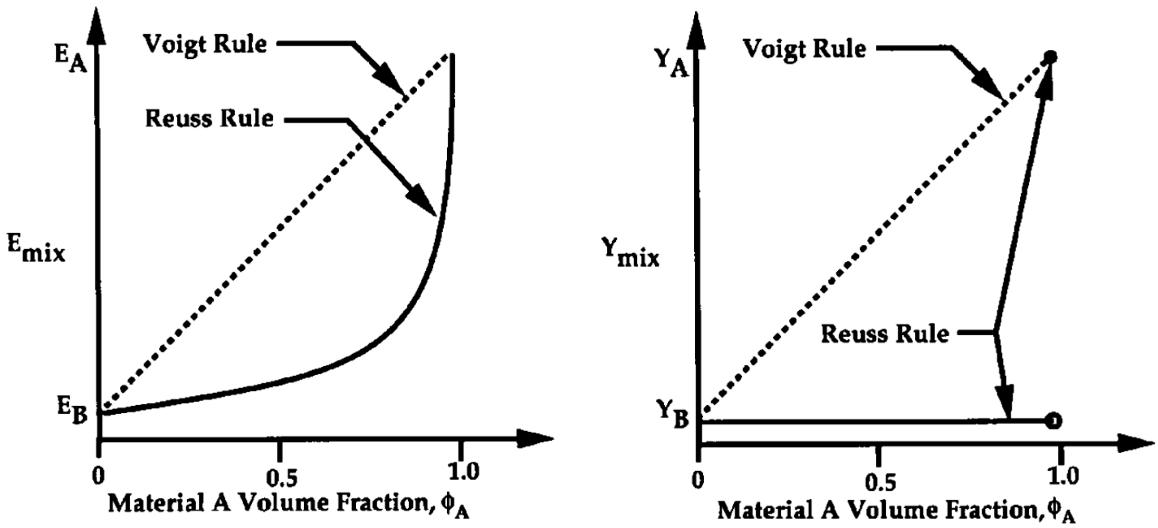

Fig. 3.4: Illustration of Young's modulus and strength referred to volume fraction φ_A of material A for Reuss and Voigt model (Swan & Kosaka, 1997)

Based in Fig. 3.4 or Eq. 3.2 and 3.4 it becomes obvious, that the Reuss and Voigt models provide the lower and upper bounds for the Young's modulus of a composite:

$$E_{Reuss} \leq E_{Composite} \leq E_{Voigt} \tag{3.5}$$

It also follows that:

$$E_{Composite} \leq \max(E_A, E_B) \tag{3.6}$$

Fig. 3.4 documents also the strength behavior for the assumption of brittle failure when stress in sping A or B reaches the ultimate limit strength. The Voigt model shows a linear relation, but the Reuss model has only two values due to the fact that the springs act in series and the weakest of the two elements determines the failure (weakest link theory).

For the hybridzed Voigt-Reuss model the following holds:

$$\varepsilon_{total} = \varphi_{Voigt} = \varepsilon_{Reuss} \quad \text{and} \quad \sigma_{total} = \alpha\sigma_{Voigt} + (1-\alpha)\sigma_{Reuss} \tag{3.7}$$

$$E_{HVR} = \alpha E_{Voigt} + (1-\alpha)E_{Reuss} \tag{3.8}$$

Where the volume fraction of the total mixture in the Voigt part is α and that in the Reuss part is (1- α). Fig. 3.5 illustrates the behaviour in comparion to the Voigt and Reuss models.

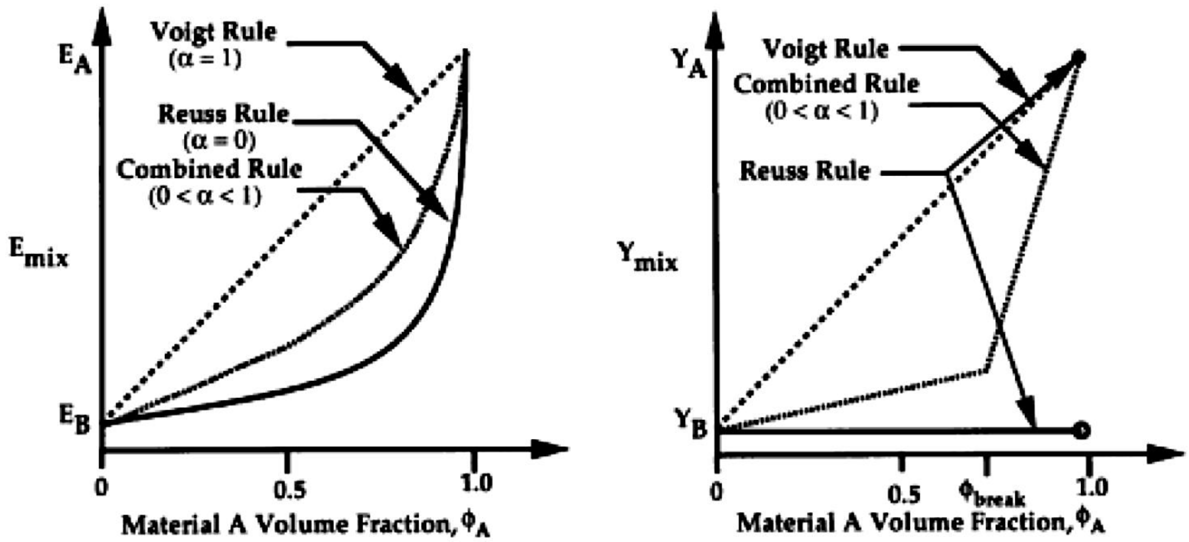

Fig. 3.5: Illustration of Young's modulus and strength referred to volume fraction φ_A of material A for Reuss and Voigt model as well as hybridized Voigt-Reuss model (Swan & Kosaka, 1997)

Based on the above discussion (Reuss and Voigt rules are upper and lower bounds) the so-called Voigt-Reuss-Hill average (VRH) is defined for the bulk moduls K_{VHR} and shear moduls G_{VHR}:

$$K_{VRH} = \frac{K_V + K_R}{2} \tag{3.9}$$

$$G_{VRH} = \frac{G_V + G_R}{2} \tag{3.10}$$

The so-called Hashin-Shtrinkman bounds provide upper and lower bounds for composite moduli for a material that macroscopically can be consired as isotropic (see Eq. 3.11 and 3.12). Upper and lower bounds are obtained by changing weighting factors (portions) φ1 and φ2. Numerical simulations confirm this behaviour as documented by Fig. 3.6.

$$K_{HS} = K_1 + \frac{\varphi_2}{(K_2 - K_1)^{-1} + \varphi_1\left(K_1 + \frac{4}{3}G_1\right)^{-1}} \tag{3.11}$$

$$G_{HS} = K_1 + \frac{\varphi_2}{(G_2 - G_1)^{-1} + \frac{2\varphi_1(K_1 + 2G_1)}{5G_1\left(K_1 + \frac{4}{3}G_1\right)}} \tag{3.12}$$

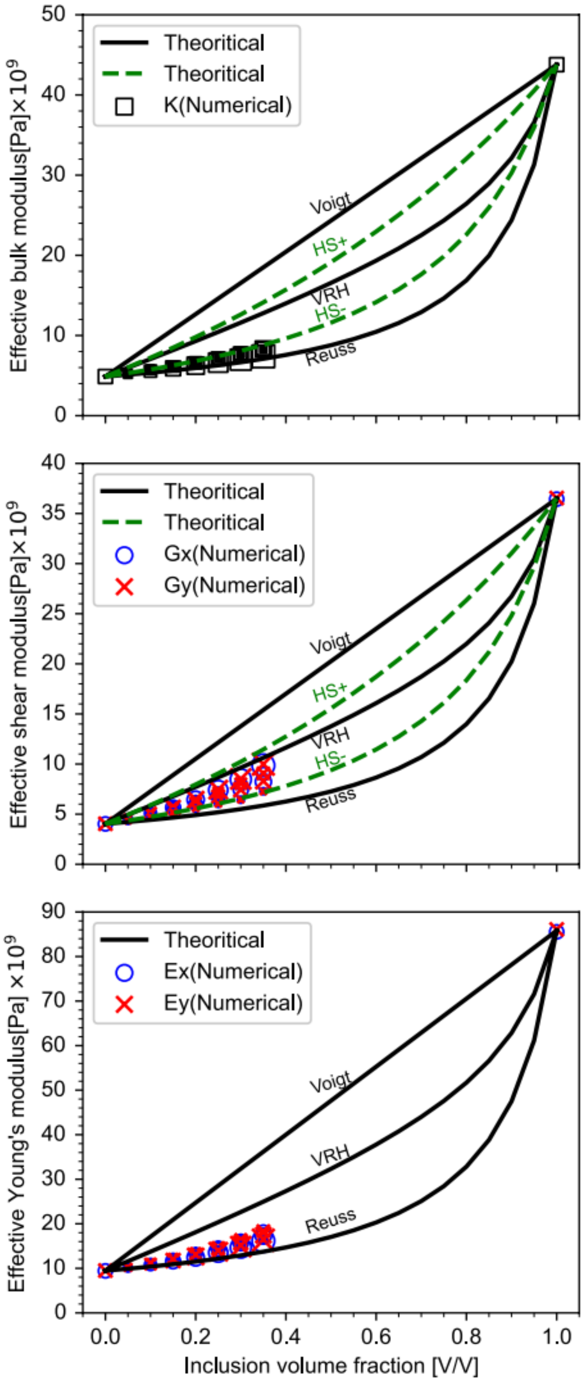

Fig. 3.6: Effective elastic moduli, comparsion between different analytical models and numerical simulations (Pothana et al., 2024)

4 Mixture rules under consideration of Possion's ratio

The consideration of Poisson's ratio υ can have significant impact on the behaviour of composits, see for instance: Liu et al. (2009) or Zhu et al. (2015). Note: by choosing special combinations of Poisson's ratio the obtained Young's modulus can be above the Voigt limit, however, it cannot exceed the modulus of the stiffest phase.

Liu et al. (2009) have derived analytical solutions for some special configuration. As shown in Fig. 4.1 a composite is compressed perpendicular to the layers A and B. The analytical solution for the resulting overall Young's modulus E_Z (z-direction) is:

$$E_Z = \frac{E_A E_B}{\varphi_A E_B + \varphi_B E_A - \frac{2\varphi_A\varphi_B\left(\upsilon_A E_B - \upsilon_B E_A\right)^2}{(1-\upsilon_A)\varphi_B E_B + (1-\upsilon_B)\varphi_A E_A}} \tag{4.1}$$

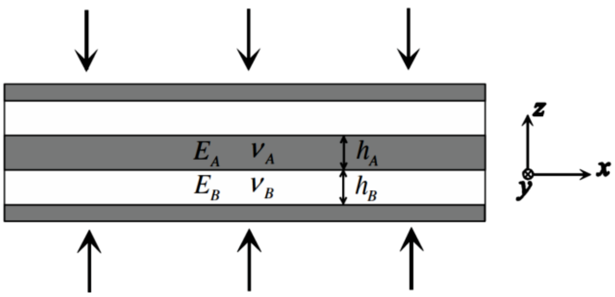

Fig. 4.1: Composite model constellation considering Poisson's ratio (Zhu et al., 2015)

As shown in Fig. 4.2 a composite is pulled parallel to the layers A and B. The analytical solution for the resulting overall Young's modulus E_x (x-direction) is:

$$E_X = \left(\varphi_A E_A + \varphi_B E_B\right) + \frac{\varphi_A\varphi_B E_A E_B\left(\upsilon_A - \upsilon_B\right)^2}{\varphi_A E_A\left(1-\upsilon_B^2\right) + \varphi_B E_B\left(1-\upsilon_A^2\right)} \tag{4.2}$$

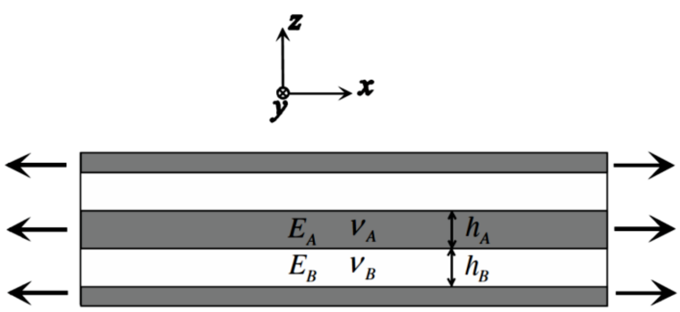

Fig. 4.2: Composite model constellation considering Poisson's ratio (Zhu et al., 2015)

5 Halpin-Tsai equation

In case the elements A and B act neither excact parallel or in series the empirical so-called Halpin-Tsai equation can be applied to estimate the overall Young's modulus:

$$E_{HT} = \frac{E_A\left[E_B + \xi\left(\varphi_B E_B + \varphi_A E_A\right)\right]}{\varphi_B E_A + \varphi_A E_B + \xi E_A} \tag{5.1}$$

Where ξ is an adjustable parameter that results in series for ξ=0 and parallel averaging for very large values of ξ.

6 Average density

The overall density ρ_{total} of a composite can be calculated by using weighted arithmetic mean:

$$\rho_{total} = \sum_{i=1}^{n} \varphi_i \rho_i \tag{6.1}$$

Where φ_i represent the volume contents of the different components.

7 Fibre reinforced materials

Fibre reinforced concrete is a widely used material in civil and geotechnical engineering. The fibres are randomly distributed inside the matrix. Therefore, a simple approximation is to consider only the volume to weight ratio between fibres and matrix as weighting function.

$$vf_m = \frac{V_m}{V_c} \quad \text{and} \quad mf_m = \frac{m_m}{m_c} \quad \text{with} \quad vf_m = 1 - vf_f \text{ and } mf_m = 1 - mf_f \tag{7.1}$$

Where vf_m = matrix volume fraction, mf_m = matrix mass fraction, m = mass, v = volume and indices c = composite, m = matrix, f = fibre.

The density ρ of a fibre reinforced material can be determined according to thes equations:

$$\rho_c = \rho_f vf_m + \rho_m mf_m \quad \text{or} \quad \rho_c = \frac{1}{\frac{mf_f}{\rho_f} + \frac{mf_m}{\rho_m}} \tag{7.2}$$

Depending on the properties several constellations can be considered. Let us first consider, that fibre and matrix have the same failure strain and brittle failure (see Fig. 7.1). The corresponding failure stresses are σ_{mu} (for matrix) and σ_{fu} (for fibres).

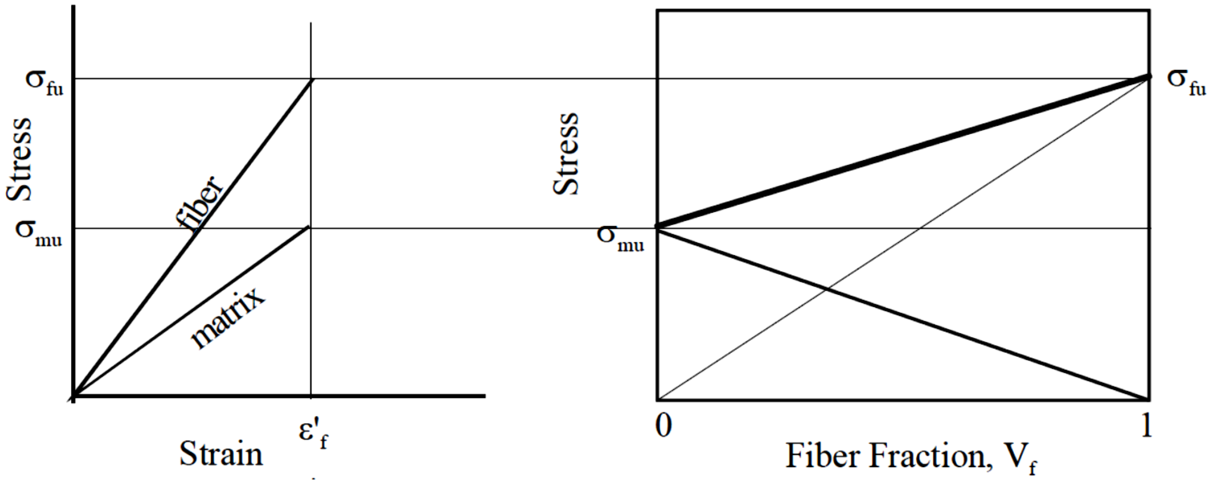

Fig. 7.1: Composite with brittle matrix and fibre as well as same failure strain (NKMe, 2021)

The failure stress of the composite σ_c can be calculated according to following equation:

$$\sigma_c = \sigma_{mu} + \left(\sigma_{fu} - \sigma_{mu}\right) vf_f \tag{7.3}$$

Let us now assume brittle fibres in a ductile matrix like illustrated in Fig. 7.2.

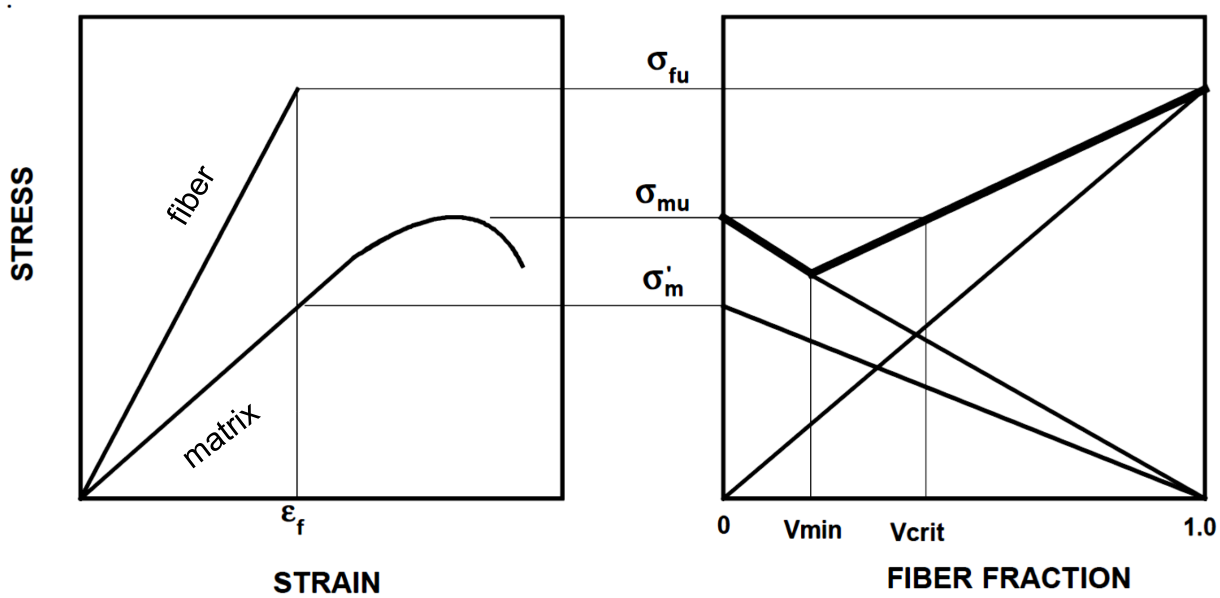

Fig. 7.2: Composite with brittle fibre and ductile matrix (NKMe, 2021)

The reduction in strength from σ_{mu} until reaching vf_f (Vmin) describes the situation where fibres break and only the remaing matrix volume can take load:

$$\sigma_c = \sigma_{mu}(1 - vf_f) \tag{7.4}$$

Beyond v_{min} the failure envelope of the composite becomes increasing:

$$\sigma_c = \sigma_{fu} vf_f + \sigma_m^l (1 - vf_f) \tag{7.5}$$

Where σ^l_m is the stress assuming failure strain ε_f of fibres multiplied with Young's modulus of pure concrete E_m.

$$\sigma_m^l = E_m \varepsilon_f \tag{7.6}$$

The portion of fibres which leads to the minimum strength of the composite (lower than the strength of the matrix alone) is given by the following equation:

$$vf_{f,\min} = \frac{\sigma_{mu} - \sigma_m^l}{\sigma_{fu} + \sigma_{mu} - \sigma_m^l} \tag{7.7}$$

Another interesting point is reached at v_{crit}, which determines the minimum amout of fibres necessary to exceed the strength of the pure matrix:

$$vf_{f,crit} = \frac{\sigma_{mu} - \sigma_m^l}{\sigma_{fu} - \sigma_m^l} \tag{7.8}$$

Let us finally consider ductile fibres imbedded in a brittle matrix, typical for steel reinforced concrete (see Fig. 7.3).

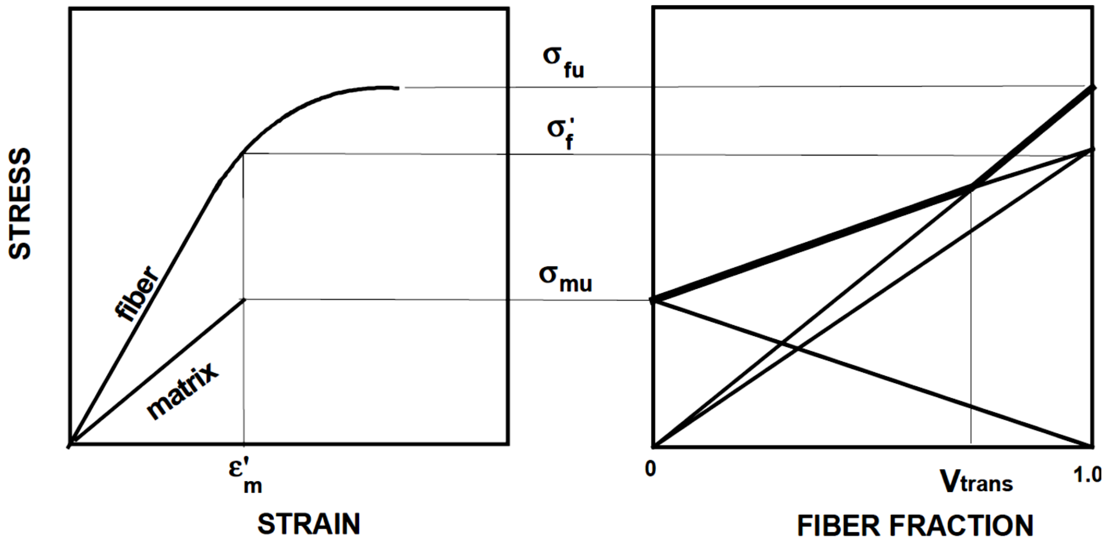

Fig. 7.3: Composite with brittle matrix and ductile fibre (NKMe, 2021)

In case the matrix fails before the fibres, the fibre stress is the following:

$$\sigma_f^l = E_f \varepsilon_m^l \tag{7.9}$$

With low proportions of fibres the composite fails when matrix fails and composite strength is given by:

$$\sigma_c = \sigma_f^l vf_f + \sigma_{mu}(1 - vf_f) \tag{7.10}$$

With high proportions of fibres the matrix can fail completely, but matrix fragments will be kept in place by fibres and composite strength is given by:

$$\sigma_c = \sigma_{fu} vf_f \tag{7.11}$$

The transition from matrix to fibre dominating failure is given by the following expression:

$$vf_{f,trans} = \frac{\sigma_{mu}}{\sigma_{fu} + \sigma_{mu} - \sigma_f^l} \tag{7.12}$$

Lee and Wang (1998) have proposed a degradation parameter P to take into account deviations from an ideal composite (non-homogeneous fibre spread and partial lack of matrix material between some adjacent fibres) to determine the tensile strength:

$$\sigma_c = \sigma_f vf_f (1 - P) + \sigma_m (1 - vf_f) \tag{7.13}$$

8 Simple examples

8.1 Characteristics of a layered roof

Fig. 8.1.1 shows a layered roof which consists of 3 layers characterized by corresponding Young's moduli E and thicknesses h. Task: Determine the overall Young's modulus parallel and perpendicular to the layering.

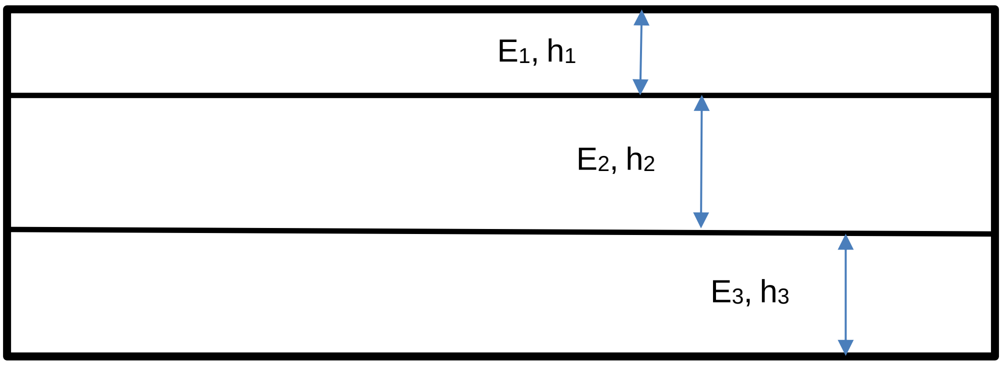

Fig. 8.1.1: Layered roof

The following values are given:

E_1 = 35 GPa h_1 = 2 m
E_2 = 15 GPa h_2 = 1 m
E_3 = 50 GPa h_3 = 5 m

$$E_{perpendicular} = \frac{h_1}{8m}E_1 + \frac{h_2}{8m}E_2 + \frac{h_3}{8m}E_3 = 41.9 \text{ GPa}$$

$$E_{parallel} = \frac{1}{\frac{h_1}{8E_1} + \frac{h_2}{8E_2} + \frac{h_3}{8E_3}} = 35.8 \text{ GPa}$$

8.2 Characterstics of a composite support element

A vertical support element consists of two material with different stiffness E und different thickness h. Task: determine the stresses inside the two components of the support and the deformation of the support element.

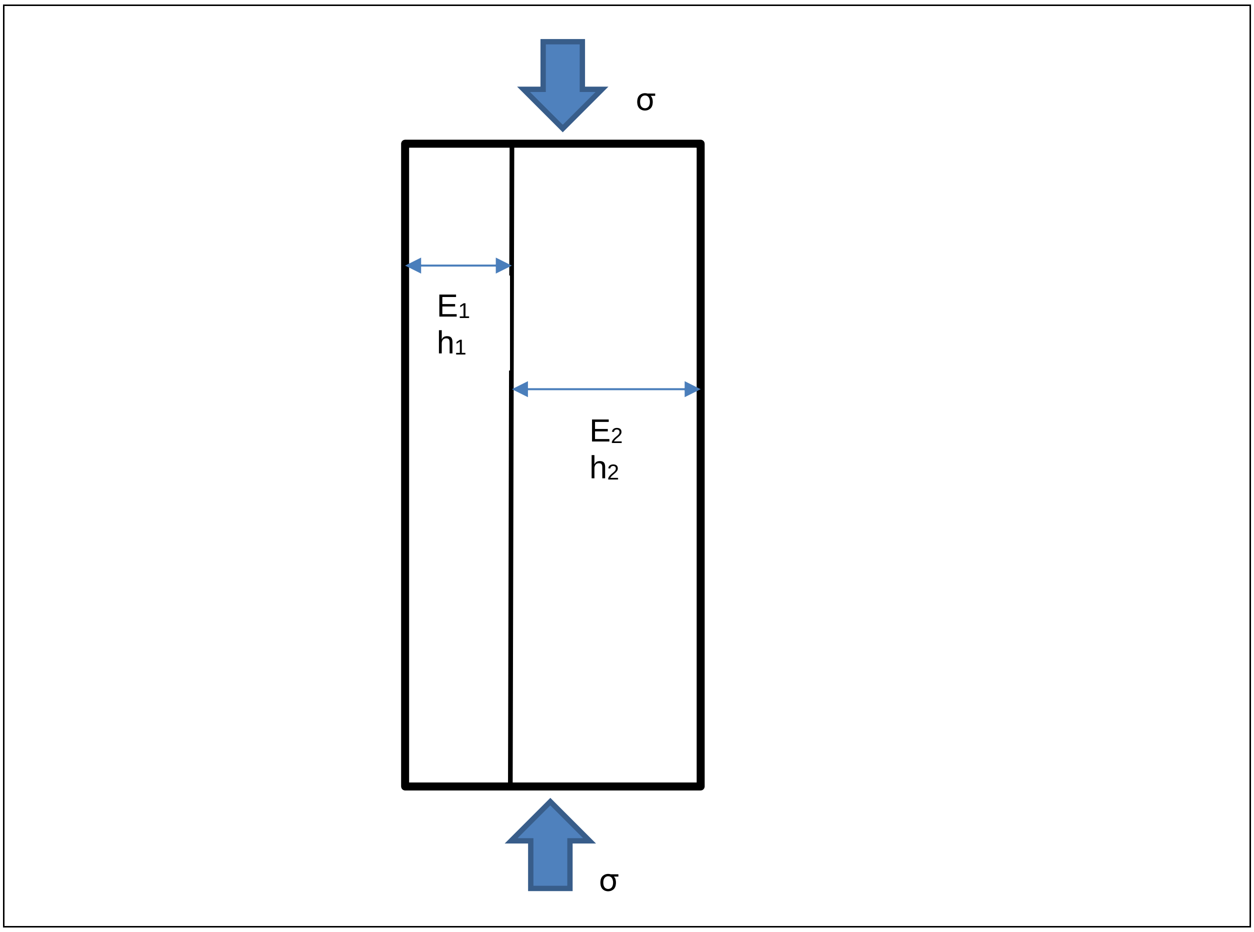

Fig. 7.2.1: Composite support element under vertical pressure

The following values are given:

E_1 = 55 GPa h_1 = 0.2 m
E_2 = 35 GPa h_2 = 0.5 m
σ = 5 MPa

$$\varepsilon = \frac{\sigma}{\frac{0.2}{0.7}E_1 + \frac{0.5}{0.7}E_2} = 1.23\text{e-}4$$

$$\sigma_1 = E_1\varepsilon = 6.8 \text{ MPa}$$

$$\sigma_2 = E_2\varepsilon = 4.3 \text{ MPa}$$

The overall Young's modulus corresponds to the Voigt model (weighted arithmetic mean):

$$E_{support} = \frac{0.2}{0.7}E_1 + \frac{0.5}{0.7}E_2 = 40.7 \text{ MPa}$$

8.3 Characteristics of fibre reinforced concrete

For a shotcret of Typ B25 (2.5 MPa tensile strength, 30 GPa Young's modulus, failure strain 0.05%) the tensile strength should be enhanced by a factor of 2 and 5 by using steel fibres (Young's modulus 210 GPa). Task: Determine the amount of steel fibres necessary to reach this goal and calculate the corresponding Young's modulus and the density of this composite.

First, according to Eq. 9 and 12, the transition point has to be determined:

$$\sigma_f^I = 210 GPa \cdot 0.0005 = 10.5 \text{ MPa}$$

$$vf_{f,trans} = \frac{2.5\, MPa}{1000\, MPa + 2.5 MPa - 10.5\, MPa} = 0.0025 = 0.25\%$$

Applying the transition point into Eq. 7.11 yields:

$$\sigma_c = 1000 MPa \cdot 0.0025 = 2.5 \text{ MPa}$$

The desired composite strength is 5 or even 12.5 MPa and has therefore to be calculated according to Eq. 7.11:

$$vf_f = \frac{\sigma_{mu}}{\sigma_{fu}} = \frac{5}{1000} = 0.005 = 0.5\% \qquad \text{for 5 MPa desired tensile strength}$$

$$vf_f = \frac{\sigma_{mu}}{\sigma_{fu}} = \frac{12.5}{1000} = 0.0125 = 1.25\% \qquad \text{for 12.5 MPa desired tensile strength}$$

To determine the corresponding Young's modulus the Halpin-Tsai equation would provide the most accurate value, however because the parameter ξ is unknown, the Reuss and Voigt approaches are used to estimate the upper and lower bound for the Young's modulus.

$$E_{\text{Re}uss} = \frac{30\, GPa \cdot 210 GPa}{0.005 \cdot 30 GPa + 0.995 \cdot 210 GPa} = 30.13 \text{ GPa} \qquad \text{for 5 MPa desired strength}$$

$$E_{\text{Re}uss} = \frac{30\, GPa \cdot 210 GPa}{0.0125 \cdot 30 GPa + 0.9875 \cdot 210 GPa} = 30.38 \text{ GPa} \qquad \text{for 5 MPa desired strength}$$

$$E_{Voigt} = 0.995 \cdot 30 GPa + 0.005 \cdot 210 GPa = 30.90 \text{ GPa} \qquad \text{for 12.5 MPa desired strength}$$

$$E_{Voigt} = 0.9875 \cdot 30 GPa + 0.0125 \cdot 210 GPa = 32.25 \text{ GPa} \qquad \text{for 12.5 MPa desired strength}$$

According to the inequality relation of the Reuss and Voigt models $E_{Reuss} \leq E_{Composite} \leq E_{Voigt}$ it holds for this specific case:

30.13 GPa ≤ $E_{Composite}$ ≤ 30.38 GPa for 5 MPa desired strength

30.90 GPa ≤ $E_{Composite}$ ≤ 32.25 GPa for 12.5 MPa desired strength

The corresponding density of the composites can be calculated according to Eq. 7.2 assuming a density for the steel fibres of 8000 kg/m^3 and for the concrete of 2000 kg/m^3.

ρ_c = 0.005·8000 + 0.995·2000 = 2030 kg/m^3 for 5 MPa desired strength

ρ_c = 0.0125·8000 + 0.9875·2000 = 2075 kg/m^3 for 5 MPa desired strength

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